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On the Nonexistence of Simplex Integration Rules for Infinite Integrals

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Abstract. It is shown that there do not exist integration rules of the form

$$\int_0^\infty f(x)\,dx = \sum_{i=1}^n w_i f(x_i) + C_n f^{(m)}(\xi), \qquad 0 < \xi < \infty.$$

Almost all classical integration rules over a finite interval are simplex, that is, they have the form

$$\int_a^b f(x)\,dx = \sum_{i=1}^n w_i f(x_i) + Cf^{(k)}(\xi), \qquad a < \xi < b, \xi = \xi(f),$$

where C is a constant depending on the rule and interval, but independent of f, and k is some integer which is characteristic for the rule. Some special rules, for example Weddle's rule, which are not simplex are multiplex, that is, the error has the form $\sum_{j=1}^m C_j f^{(k_j)}(\xi_j)$. It is the aim of this note to show that there can exist no simplex or multiplex rule for the infinite integral $\int_0^\infty f(x)\,dx$. Although the Gauss-Laguerre rule

$$\int_0^\infty e^{-x} f(x)\,dx = \sum_{i=1}^n w_i f(x_i) + C_n f^{(2n)}(\xi)$$

appears to have the form of a simplex rule, this is not so, since we are concerned with unweighted integrals and if we write $f(x) = e^{-x}e^x f(x)$, we have that

$$\int_0^\infty f(x)\,dx = \sum_{i=1}^n w_i e^{x_i} f(x_i) + C_n (e^x f(x))^{(2n)}_{x=\xi}$$

which is neither simplex nor multiplex in form.

We now show that it is impossible to have an integration rule of the form

$$(1) \qquad If \equiv \int_0^\infty f(x)\,dx = \sum_{i=1}^n w_i f(x_i) + Cf^{(k)}(\xi), \qquad 0 < \xi < \infty,$$

valid for all $f \in L[0, \infty) \cap C^k(0, \infty)$, or for that matter, one where there are a finite number of terms of the form $C_j f^{(k_j)}(\xi_j)$. The proof is based on the simple fact that, for any $r > 0$,

$$(2) \qquad \int_0^\infty f(x)\,dx = r \int_0^\infty f(rx)\,dx.$$

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If (1) were true, then (2) would imply that

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\begin{aligned}
If &= r\int_0^\infty f(rx)\,dx = r\sum_{i=1}^{n} w_i f(rx_i) + rCf^{(k)}(r\xi) \\
&= r\sum_{i=1}^{n} w_i f(rx_i) + r^{k+1}Cf^{(k)}(\xi), \qquad 0 < \xi < \infty,
\end{aligned}
\tag{3}
$$

which must hold for all $f \in L[0, \infty) \cap C^k(0, \infty)$ and any real r. If we now choose such a function which is bounded together with its kth derivative on $[0, \infty)$, say $f(x) = 1/(1 + x^2)$, and let r approach zero, we see that the right-hand side of (3) approaches zero while the left-hand side has a constant value. This contradiction proves our result.

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